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DEMONSTRATING SHRODENGER EQUATION INVOLVING HARMONIC OSCILLATOR POTENTIAL WITH A POSITION DEPENDENT MASS IN AN EXTERNAL ELECTRIC FIELD

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Abstract. An external electric field is provided to the familiar one-dimensional quantum harmonic oscillator model with a various position mass $m(x) = \frac{a\, m_0}{a+x}$. The Nikiforov-Uvarov approach is involved to investigate a particular solution to the exact Schrödinger equation. The exactly solvable confined model of the quantum harmonic oscillator in an external electric field was proposed. Starting with the BenDanial-Duke kinetic energy operator approximations, the construction of the position-dependent mass Schrödinger equation is studied. The analytic representation of the wave functions of the stationary states is expressed analytically and graphically using modified Laguerre polynomials as well as the energy spectrum. In contrast with the absence of an external electric field, when the energy spectrum totally overlaps with that of the harmonic oscillator potential in an external electric field: the energy spectrum becomes non-equidistant and varies depending on some factors. The Nikiforov-Uvarov approach succeeds broadly in demonstrating the wave function and energy spectrum and showing good sense.

Keywords: *Schrödinger equation, Nikiforov-Uvarov method, Quantum harmonic oscillator potential, position dependent mass.*

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1. Introduction

A quantum mechanical equation called the Position dependent mass (PDM) Schrödinger Equation (SE) shows how a particle with variable mass behaves while moving through a potential energy field. The mass of the particle is supposed to be constant in the classical Schrödinger equation. However, some physical systems, particularly those related to condensed matter physics and nanotechnology, show PDM distributions.

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In the PDM SE, the particle's mass is allowed to change with position. Compared to the original SE, this adds an additional level of complexity to the equation. The equation considers both the particle's spatial variation in mass as well as the potential energy landscape it is traversing. Novel quantum mechanical effects and behaviors that do not exist in systems with constant mass may result from this.

The common form of this equation is:

$$-\frac{\hbar^2}{2}\nabla.\left(\frac{1}{\mathrm{m(x)}}\nabla\psi(\mathrm{x})\right)+\mathrm{V(x)}\psi(\mathrm{x})=\mathrm{E}\psi(\mathrm{x}), \tag{1}$$

where $\hbar$ is the reduced Planck constant, ∇ is the gradient operator and $\mathrm{m(x)}$ is the PDM.

In recent years, several attractive research have been conducted on quantum systems with effective PDM (von Roos, 1983; Smith *et al.*, 1990; Barranco *et al.*, 1997). In many applied areas such as modern physics, the Schrödinger equations with a position-dependent mass are a highly helpful model. The research of the electronic properties of semiconductors (Bastard, 1990; Weisbuch *et al.*, 1993), graded alloys and semiconductor hetero structures (Lévai, 1994), quantum liquids (De Saavedra, 1994), quantum wells and quantum dots (Harrison *et al.* ,2016; Galbraith et *al.*, 1988; Young, 1983) are examples of special applications in condensed matter.

The rigorous solutions of the SE defining certain nonrelativistic quantum systems are always appealing due to their enormous potential for explaining a broad variety of occurrences in quantum physics and attached fields. Among such interesting quantum mechanical difficulties, another challenge is provided by involving an external field to a quantum system.

This work uses a transformation of the wave function to investigate the effective PDM SE; to solve the effective PDM SE for harmonic oscillator potential in the presence of an external electric field for first time in our known, where we used the Nikiforov-Uvarov (NU) approach.

The following is the outline for the present paper. The PDM SE by using the BenDanial-Duke kinetic energy operator is reviewed in Section 2. The NU method is outlined in Section 3. The effective PDM SE is solved in Section 4. A summary concludes the paper.

2. PDM SE Using BenDaniel-Duke Kinetic Energy Operator

For our calculations, we choose the Hermitian form of the kinetic energy operator with effective PDM (also known as the BenDaniel-Duke kinetic energy operator) (BenDaniel, 1966).

Starting with BenDaniel-Duke kinetic energy operator:

$$\widehat{\mathrm{H}}_0^{\mathrm{BD}}=-\frac{\hbar^2}{2}\frac{\mathrm{d}}{\mathrm{dx}}\frac{1}{\mathrm{m(x)}}\frac{\mathrm{d}}{\mathrm{dx}}, \tag{2}$$

$$\widehat{\mathrm{H}}_0^{\mathrm{BD}}=-\frac{\hbar^2}{2}\frac{\mathrm{d}}{\mathrm{dx}}\left[\frac{1}{\mathrm{m(x)}}\frac{\mathrm{d}}{\mathrm{dx}}\right]. \tag{3}$$

Now we have multiplication derivatives.

$$\widehat{H}_0^{BD} = -\frac{\hbar^2}{2}\left[-\frac{m'(x)}{m^2(x)}\frac{d}{dx} + \frac{1}{m(x)}\frac{d^2}{dx^2}\right]. \tag{4}$$

Taking $1/m(x)$ as a common factor:

$$\widehat{H}_0^{BD} = -\frac{\hbar^2}{2m(x)}\left[\frac{d^2}{dx^2} - \frac{m'(x)}{m(x)}\frac{d}{dx}\right]. \tag{5}$$

BenDaniel–Duke Hamiltonian is given by using equation (4) and the potential energy:

$$\widehat{H}^{BD} = \widehat{H}_0^{BD} + V(x). \tag{6}$$

Using (5) and (6), we get

$$\widehat{H}^{BD} = -\frac{\hbar^2}{2m(x)}\left(\frac{d^2}{dx^2} - \frac{m'(x)}{m(x)}\frac{d}{dx}\right) + V(x). \tag{7}$$

Schrödinger equation when using the BenDaniel–Duke Hamiltonian becomes as

$$\widehat{H}^{BD}\psi(x) = E\psi(x). \tag{8}$$

Using equation (7) in (8), we get

$$\left[-\frac{\hbar^2}{2m(x)}\left(\frac{d^2}{dx^2} - \frac{m'(x)}{m(x)}\frac{d}{dx}\right) + (V(x) - E)\right]\psi(x) = 0. \tag{9}$$

The last equation is called the position-dependent mass Schrödinger equation, which results from using the position-dependent mass BenDaniel–Duke kinetic energy operator.

3. Nikiforov Uvarov (NU) Method

The NU method, commonly known as the (NU) technique, is a mathematical approach for solving second-order linear differential equations. This method is effective for solving Schrödinger-type equations in quantum mechanics (Al-Hawamdeh *et al.*, 2023), particularly those involving quantum systems with special forms of potentials.

The method involves reducing the second-order differential equation to a known orthogonal polynomial equation, such as the Hermite, Laguerre or Jacobi equation. The mathematical features of these orthogonal polynomials are well defined and solving the resulting orthogonal polynomial equation allows one to identify the solutions to the original differential equation.

The Schrödinger equation in one dimension is simplified using this technique for a given potential to a generalized hypergeometric equation with the necessary, coordinate transformation $s = s(x)$. and it can be written as follows:

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0, \tag{10}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials of degree at most two, $\tilde{\tau}(s)$ is 1st degree polynomial, and $\psi(s)$ is a hyper geometric function.

And the SE is written for any potential in the general form shown below:

$$\left[\frac{d^2}{ds^2}+\frac{\alpha_1-\alpha_2 s}{s(1-\alpha_3 s)}\frac{d}{ds}+\frac{-\xi_1 s^2+\xi_2 s-\xi_3}{s^2(1-s)^2}\right]\psi=0. \tag{11}$$

To get a particular solution for equation (10) by separation of variables, the wave function is made up of multiples of two separate sections as:

$$\psi(s)= \varphi(s)y(s). \tag{12}$$

If one deals with the above transformation; then equation (10) modified to the hyper geometric equation:

$$\sigma(s)y_n''(s)+\tau(s)\,y_n'(s)+\lambda y_n(s)=0, \tag{13}$$

where

$$\tau(s)=2\pi(s)+\tilde{\tau}(s). \tag{14}$$

Its derivative is negative $\tau'(s)<0$ (Nikiforov & Vasiliĭ, 1988), this condition helps to generate physical solutions.

And $\varphi(s)$ is described as a derivative of a logarithm:

$$\frac{\varphi'(s)}{\varphi(s)}=\frac{\pi(s)}{\sigma(s)}, \tag{15}$$

here, $\pi(s)$ is a polynomial with one degree or less.

Equation (13) is a hypergeometric-type differential equation and its solution is given by Rodrigues relation (Nikiforov & Uvarov, 1988).

$$y_n(s)=\frac{B_n}{\rho(s)}\frac{d^n}{ds^n}[\sigma^n(s)\rho(s)], \tag{16}$$

where B_n is the normalization constant, $\rho(s)$ is the weight function, and n is a fixed given number.

The weight function $\rho(s)$ is satisfies the following differential equation.

$$\frac{d}{ds}[\sigma(s)\rho(s)]=\tau(s)\rho(s), \tag{17}$$

or

$$\frac{\rho'(s)}{\rho(s)}=\frac{\tau(s)-\sigma'(s)}{\sigma(s)}. \tag{18}$$

The function of $\pi(s)$ is given by

$$\pi(s)=\frac{\sigma'-\tilde{\tau}(s)}{2}\pm\sqrt{\left(\frac{\sigma'-\tilde{\tau}(s)}{2}\right)^2-\tilde{\sigma}(s)+k\sigma(s)} \tag{19}$$

and

$$k = \lambda - \pi'(s). \tag{20}$$

As a result, setting the discriminated of the square root in (19) to zero for the computation of $\pi(s)$ makes the finding of k the most important step. Additionally, the eigenvalue equation described in (20) now has the new form:

$$\lambda = \lambda_n = -n\,\tau' - \frac{n(n-1)}{2}\sigma''(s)\,,(n = 0, 1, 2, \dots). \tag{21}$$

Prime factors, in this case, signify the first-degree differentials.

4. Harmonic oscillator potential (HOP) with a PDM under external electric field

A fundamental idea in quantum physics is that HOP describes the fundamental behavior of particles inside confining potentials. The complexity of quantum systems greatly increases when the effect of an external electric field and the consideration of a PDM are coupled. To fully comprehend the resulting quantum dynamics, this work concerned to several branches: the interesting interaction between a HOP; PDM and an external electric field.

The potential energy function is a confined harmonic oscillator potential in an external electric field as follows:

$$V(x) = V^{ho}(x) + V^{ext}(x). \tag{22}$$

Then

$$V(x) = \left\{\frac{m(x)\omega^2 x^2}{2} - q\mathcal{E}x\,, -a < x < +\infty\right\}, \tag{23}$$

where $m(x)$ is the mass function, ω is the frequency of the oscillator, q is the charge of electron, $\mathcal{E}$ is the electric field and a is the confinement parameter.

Using equation (23) into equation (7), we get

$$\widehat{H}^{BD} = -\frac{\hbar^2}{2m(x)}\left(\frac{d^2}{dx^2} - \frac{m'(x)}{m(x)}\frac{d}{dx}\right) + \left(\frac{m(x)\omega^2 x^2}{2} - q\mathcal{E}x\right). \tag{24}$$

Then the Schrödinger equation when using the BenDaniel–Duke Hamiltonian, by substituting equation (24) in (8), we get

$$\left[-\frac{\hbar^2}{2m(x)}\left(\frac{d^2}{dx^2} - \frac{m'(x)}{m(x)}\frac{d}{dx}\right) + \left(\frac{m(x)\omega^2 x^2}{2} - q\mathcal{E}x - E\right)\right]\psi(x) = 0. \tag{25}$$

Multiplying both sides by $\left(-\frac{2m(x)}{\hbar^2}\right)$, so it becomes

$$\left[\frac{d^2}{dx^2} - \frac{m'(x)}{m(x)}\frac{d}{dx} - \frac{2m(x)}{\hbar^2}\left(\frac{m(x)\omega^2 x^2}{2} - q\mathcal{E}x - E\right)\right]\psi(x) = 0. \tag{26}$$

More simplifying

$$\left[\frac{d^2}{dx^2}-\frac{m'(x)}{m(x)}\frac{d}{dx}-(\frac{m^2(x)\omega^2}{\hbar^2}x^2-\frac{2m(x)q\mathcal{E}}{\hbar^2}x-\frac{2m(x)E}{\hbar^2})\right]\psi(x)=0. \tag{27}$$

To satisfy the confinement effect of the potential (23), we must now define effective PDM. So, we demand that the effective PDM $m(x)$ meet the following requirements: $m(x)$ should return the accurate constant mass m_0 at the origin of position $x=0$ and afterwards within the bounds of $a\to\infty$, At the position $x=-a$, the HOP should have an infinite high wall, or $V(x)=+\infty$ for $-\infty<x\leq a$ and it should be possible to precisely solve the SE for the Hamiltonian with the BenDaniel-Duke kinetic energy operator and HOP with the PDM $m(x)$, i.e., with analytic expressions for the stationary state wave functions and the energy spectrum. Now we define the following analytical formula for the effective PDM based on the conditions given above, as

$$m(x)=\frac{a\,m_0}{a+x}, \tag{28}$$

when $m(0)=m_0$ then the fact that the first condition is easily satisfied as well as

$$\lim_{a\to\infty}\frac{a\,m_0}{a+x}=m_0. \tag{29}$$

Regarding the second requirement, we note that the potential (23) with effective PDM $m(x)$ (28) has the appropriate infinite high walls boundary conditions.

$$V(-a)=+\infty. \tag{30}$$

The only thing left to do is to demonstrate that the third criterion has been satisfied. As a result, we must explicitly solve the SE for the Hamiltonian $\widehat{H}^{BD}$ (7). Considering that

$$m'(x)=-\frac{a\,m_0}{(a+x)^2} \tag{31}$$

and

$$\frac{m'(x)}{m(x)}=-\frac{1}{a+x}. \tag{32}$$

Now substituting the equations (28) and (32), in equation (27). So, we obtained the Schrödinger equation describing the motion of our oscillator model in an external electric field.

$$\left[\begin{array}{c}\frac{d^2}{dx^2}+\frac{1}{a+x}\frac{d}{dx}-\\ \left(\frac{\frac{a^2m_0^2\omega^2}{\hbar^2}x^2-\frac{2a\,m_0q\mathcal{E}}{\hbar^2}x(a+x)-\frac{2a\,m_0E}{\hbar^2}(a+x)}{(a+x)^2}\right)\end{array}\right]\psi(x)=0. \tag{33}$$

Now expanding the above equation so it becomes as

$$\left[\frac{d^2}{dx^2}+\frac{1}{a+x}\frac{d}{dx}-\frac{\left(\frac{a^2m_0^2\omega^2}{\hbar^2}x^2-\frac{2a^2m_0q\mathcal{E}}{\hbar^2}x-\frac{2a\,m_0q\mathcal{E}}{\hbar^2}x^2-\frac{2a^2m_0E}{\hbar^2}-\frac{2a\,m_0E}{\hbar^2}x\right)}{(a+x)^2}\right]\psi(x)=0 \qquad (34)$$

Now we want to introduce a new variable so that we can solve the above equation, this variable is ξ and it’s a dimensionless variable as

$$\xi=\frac{x}{a} \qquad (35)$$

with

$$\frac{d}{dx}=\frac{1}{a}\frac{d}{d\xi} \qquad (36)$$

and taking the square for both sides

$$\frac{d^2}{dx^2}=\frac{1}{a^2}\frac{d^2}{d\xi^2}. \qquad (37)$$

After that the equation becomes

$$\left[\frac{d^2}{d\xi^2}+\frac{1}{1+\xi}\frac{d}{d\xi}-\frac{\left(\frac{a^4m_0^2\omega^2}{\hbar^2}\xi^2-\frac{2a^3m_0q\mathcal{E}}{\hbar^2}\xi-\frac{2a^3\,m_0q\mathcal{E}}{\hbar^2}\xi^2-\frac{2a^2m_0E}{\hbar^2}-\frac{2a^2\,m_0E}{\hbar^2}\xi\right)}{(1+\xi)^2}\right]\psi(\xi)=0 \qquad (38)$$

with

$$\alpha_0=\frac{2a^2m_0E}{\hbar^2} \qquad (39)$$

$$\alpha_1=\frac{2a^3m_0q\mathcal{E}}{\hbar^2} \qquad (40)$$

and

$$\alpha_2=\frac{a^4m_0^2\omega^2}{\hbar^2}-\alpha_0. \qquad (41)$$

Introducing this notation after that

$$\lambda_0=\sqrt{\frac{m_0\omega}{\hbar}}. \qquad (42)$$

Equation (40) becomes as

$$\alpha_2=a^4\lambda_0^4-\alpha_0. \qquad (43)$$

Substitute equations (39), (40) and (41) in equation (38) it becomes as

$$\left[\frac{d^2}{d\xi^2}+\frac{1}{1+\xi}\frac{d}{d\xi}+\frac{\alpha_0\ +(\alpha_0+\alpha_1)\xi-(\alpha_0-\ \alpha_1+\alpha_2)\xi^2}{(1+\xi)^2}\right]\psi(\xi)=0. \tag{44}$$

In another way

$$\psi''+\frac{1}{1+\xi}\psi'+\frac{\alpha_0\ +(\alpha_0+\alpha_1)\,\xi-(\alpha_0-\ \alpha_1+\alpha_2)\xi^2}{(1+\xi)^2}\psi=0, \tag{45}$$

where

$$\psi''=\frac{d^2\psi}{d\xi^2}\,,\qquad \psi'=\frac{d\psi}{d\xi}. \tag{46}$$

Now we are going to apply NU method to solve equation (45), so the second order differential equations of this type:

$$\psi''+\frac{\tilde{\tau}}{\sigma}\psi'+\frac{\tilde{\sigma}}{\sigma^2}\psi=0 \tag{47}$$

can be solved using this technique.

Here

$$\tilde{\tau}=1, \tag{48}$$

$$\sigma=1+\xi, \tag{49}$$

$$\tilde{\sigma}=\alpha_0\ +(\alpha_0+\alpha_1)\,\xi-(\alpha_0-\ \alpha_1+\alpha_2)\xi^2. \tag{50}$$

Now we search for the answer to equation (47) by introducing $\psi(\xi)$ as follows:

$$\psi(\xi)=\varphi(\xi)y, \tag{51}$$

where $\varphi(\xi)$ represented in terms of σ and π which is an arbitrary polynomial essentially of first degree.

In this context, $\varphi(\xi)$ is defined in the following way (Mammadova, 2022):

$$\varphi(\xi)=e^{\int\frac{\pi(\xi)}{\sigma(\xi)}d\xi}, \tag{52}$$

where $\pi(\xi)$ is also a polynomial at least to the 1st degree.

Then equation (51), becomes as

$$\psi(\xi)=e^{\int\frac{\pi(\xi)}{\sigma(\xi)}d\xi}y. \tag{53}$$

Simple calculations make it possible to find

$$\psi'(\xi)=\left(\frac{\sigma(\xi)y'+\pi(\xi)y}{\sigma}\right)e^{\int\frac{\pi(\xi)}{\sigma(\xi)}d\xi} \tag{54}$$

and the second derivative is

$$\psi''(\xi)\ =\left[\frac{\sigma(\xi)(\sigma(\xi)y''+2\pi(\xi)y')+y\big(\sigma(\xi)\pi'(\xi)-\pi(\xi)\sigma'(\xi)+\pi^2(\xi)\big)}{\sigma^2(\xi)}\right]e^{\int\frac{\pi(\xi)}{\sigma(\xi)}d\xi}. \tag{55}$$

Equation (55) in another way

$$\psi''(\xi)=\left[\frac{\sigma(\xi)\pi'(\xi)-\pi(\xi)\sigma'(\xi)+\pi^2(\xi)}{\sigma^2(\xi)}y+\frac{2\pi(\xi)}{\sigma(\xi)}y'+y''\right]e^{\int\frac{\pi(\xi)}{\sigma(\xi)}d\xi}. \tag{56}$$

These calculations leas to the equation for $y(\xi)$

$$y''+\frac{2\pi+\tilde{\tau}}{\sigma}y'+\frac{\tilde{\sigma}+\pi^2+\pi\,(\tilde{\tau}-\sigma')+\pi'\sigma}{\sigma^2}y=0, \tag{57}$$

where

$$\tau=2\pi+\tilde{\tau} \tag{58}$$

and

$$\overline{\sigma}=\tilde{\sigma}+\pi^2+\pi\,(\tilde{\tau}-\sigma')+\pi'\sigma. \tag{59}$$

So, we can write equation (57) as

$$y''+\frac{\tilde{\tau}}{\sigma}y'+\frac{\tilde{\sigma}}{\sigma^2}y=0. \tag{60}$$

Concerning that $\overline{\sigma}$ is a polynomial of mostly 2nd degree, it is necessary that

$$\overline{\sigma}=\lambda\sigma \tag{61}$$

with $\lambda=\text{const}$.

If we put equation (61) in equation (59), so it becomes as

$$\pi^2+\pi\,(\tilde{\tau}-\sigma')+\tilde{\sigma}-(\lambda-\pi')\sigma=0. \tag{62}$$

Introducing a new notation.

$$k=\lambda-\pi' \tag{63}$$

Equation (62), becomes as quadratic equation for $\pi(\xi)$

$$\pi^2+\pi\,(\tilde{\tau}-\sigma')+\tilde{\sigma}-k\sigma=0. \tag{64}$$

Let

$$\tilde{\tau}-\sigma'=0. \tag{65}$$

Then quadratic equation (64) simplified as

$$\pi^2+\tilde{\sigma}-k\sigma=0. \tag{66}$$

Thus

$$\pi=\mu\sqrt{k\sigma-\tilde{\sigma}}, \tag{67}$$

where $\mu=\pm1$.

Now the following possible expressions for $\pi(\xi)$ are then obtained.

$$\pi(\xi)=\begin{Bmatrix}\mu\sqrt{-\alpha_1\,\xi+(\alpha_0-\ \alpha_1+\alpha_2)\xi^2}\,, & k=\alpha_0\\ \mu\sqrt{(a^4\lambda_0^4-\alpha_1)}\ \xi, k=\alpha_0+\alpha_1 & \end{Bmatrix}. \tag{68}$$

Now, after discovering the explicit expressions of the functions $(\pi(\xi))$ and $(\sigma(\xi))$, one can also get the explicit expression of $(\varphi(\xi))$ from (47), after using simple calculations and for this case $k=\alpha_0+\alpha_1$ we obtain

$$\varphi(\xi) = (1+\xi)^{-\mu a^2\lambda_0^2}\ e^{\mu\sqrt{a^4\lambda_0^4-\alpha_1}\ \xi}. \tag{69}$$

For $\varphi(\xi)$, the essential boundary conditions are fulfilled:

$$\lim_{\xi\to-1}\varphi(\xi) = 0\,, \quad \lim_{\xi\to\infty}\varphi(\xi) = 0. \tag{70}$$

To satisfy the requirement $\mu = -1$ for $\varphi(\xi)$, the case $k = \alpha_0 + \alpha_1$ should be chosen. These results result in the final expressions of $\pi(\xi)$ and $\varphi(\xi)$ as follows:

$$\pi(\xi) = \mu\sqrt{(a^4\lambda_0^4 - \alpha_1)}, \tag{71}$$

$$\varphi(\xi) = (1+\xi)^{a^2\lambda_0^2}\ e^{-\sqrt{a^4\lambda_0^4-\alpha_1}\ \xi}. \tag{72}$$

Now equation (51) becomes as

$$\psi(\xi) = \left[(1+\xi)^{a^2\lambda_0^2}\ e^{-\sqrt{a^4\lambda_0^4-\alpha_1}\ \xi}\right] y. \tag{73}$$

The 2nd order differential equation for y that demonstrated from substituting ψ in equation (45) is as follows:

$$\begin{aligned}(1+\xi)y'' + \left(2a^2\lambda_0^2 + 1 - 2\sqrt{a^4\lambda_0^4 - \alpha_1}\,(1+\xi)\right)y' \\ = \left((2a^2\lambda_0^2 + 1)\sqrt{a^4\lambda_0^4 - \alpha_1} - 2a^4\lambda_0^4 + \alpha_1 - \alpha_0\right)y.\end{aligned} \tag{74}$$

Now compare the above equation with the following equation for the generalized Laguerre polynomials to arrive at polynomial solutions (Koekoek *et al*., 2010)

$$(x-d)y_n''(c) + \{2\varepsilon(x-d) + \alpha + 1\}y_n'(x) = 2\varepsilon\, n\, y_n(x), \tag{75}$$

where $d < x, \varepsilon < 0$ and $\alpha + 1 > 0$.

Thus

$$y_n(x) = L_n^{\alpha}\left(2\varepsilon\,(d-x)\right). \tag{76}$$

Here the first four values of generalized Laguerre polynomials

Table 1. The first four values of generalized Laguerre polynomials

n	$L_n^{\alpha}(x)$
0	1
1	$-x + (\alpha+1)$
2	$\frac{x^2}{2} - (\alpha+2)x + \frac{(\alpha+1)(\alpha+2)}{2}$
3	$-\frac{x^3}{6} + \frac{(\alpha+3)}{2}x^2 - \frac{(\alpha+2)(\alpha+3)}{2}x + \frac{(\alpha+1)(\alpha+2)(\alpha+3)}{6}$

After comparing equation (74) with (75) we get.

$$y_n(x) = L_n^{2a^2\lambda_0^2}\left(2\sqrt{a^4\lambda_0^4 - \alpha_1}\ \left(1 + \frac{x}{a}\right)\right). \tag{77}$$

Using equation (40) and equation (42)

$$y_n(x) = L_n^{(2\frac{m_0\omega}{\hbar}a^2)} \left(2\frac{m_0\omega}{\hbar} a \sqrt{1-\frac{2q\mathcal{E}}{m_0\omega^2 a}}\ (1+\frac{x}{a})\right) \tag{78}$$

and put

$$\alpha = 2\frac{m_0\omega}{\hbar}a^2. \tag{79}$$

The solution to this second-order differential equation for $y_n(x)$ is already known. Comparing (74) with (75),

$$\left(2a^2\lambda_0^2\sqrt{a^4\lambda_0^4-\alpha_1}+\sqrt{a^4\lambda_0^4-\alpha_1}-2a^4\lambda_0^4+\alpha_1-\alpha_0\right) = -2n\sqrt{a^4\lambda_0^4-\alpha_1}\,, \tag{80}$$

we get

$$\left(2a^2\lambda_0^2\frac{a^2 m_0\omega}{\hbar}\sqrt{1-\frac{2q\mathcal{E}}{m_0\omega^2 a}}+\frac{a^2 m_0\omega}{\hbar}\sqrt{1-\frac{2q\mathcal{E}}{m_0\omega^2 a}}-2\frac{a^4 m_0^2\omega^2}{\hbar^2} + \frac{2a^3 m_0 q\mathcal{E}}{\hbar^2}-\frac{2a^2 m_0 E}{\hbar^2}\right) = -2n\frac{a^2 m_0\omega}{\hbar}\sqrt{1-\frac{2q\mathcal{E}}{m_0\omega^2 a}}. \tag{81}$$

Then the energy eigen value equation is

$$E = \left(a^2\lambda_0^2\hbar\omega\sqrt{1-\frac{2q\mathcal{E}}{m_0\omega^2 a}}+\frac{\hbar\omega}{2}\sqrt{1-\frac{2q\mathcal{E}}{m_0\omega^2 a}}-m_0a^2\omega^2+q\mathcal{E} + n\hbar\omega\sqrt{1-\frac{2q\mathcal{E}}{m_0\omega^2 a}}\right). \tag{82}$$

Combining the same terms now

$$E = E_n^{q\mathcal{E}} = \hbar\omega\sqrt{1-\frac{2q\mathcal{E}}{m_0\omega^2 a}}\left(n+\frac{1}{2}+a^2\frac{m_0\omega}{\hbar}\right)-m_0a^2\omega^2+q\mathcal{E}. \tag{83}$$

After these computations and by using equations (73) and (78) we get the orthonormal wave functions, which presented by a particular exact formula:

$$\psi_n^{q\mathcal{E}}(x) = C_n^{q\mathcal{E}}\left(1+\frac{x}{a}\right)^{\frac{m_0\omega}{\hbar}a^2} e^{-\frac{m_0\omega}{\hbar}a\sqrt{1-\frac{2q\mathcal{E}}{m_0\omega^2 a}}\,(x+a)} L_n^{(2\frac{m_0\omega}{\hbar}a^2)}\left(2\frac{m_0\omega}{\hbar}a\sqrt{1-\frac{2q\mathcal{E}}{m_0\omega^2 a}}\ (x+a)\right). \tag{84}$$

Now, the exact form of the normalization constant of PDM SE with the HOP. Normalization constant can be derived from the wave functions' orthogonality relation.

$$\int_{-\infty}^{\infty}\tilde{\psi}_m(x)\tilde{\psi}_n(x)\,dx = \int_{-a}^{\infty}\tilde{\psi}_m(x)\tilde{\psi}_n(x)\,dx = \delta_{mn}. \tag{85}$$

Here we get

$$C_n^{q\mathcal{E}} = (-1)^n\left(2\frac{m_0\omega}{\hbar}a^2\sqrt{1-\frac{2q\mathcal{E}}{m_0\omega^2 a}}\right)^{\frac{m_0\omega}{\hbar}a^2+\frac{1}{2}}\sqrt{\frac{n!}{a\Gamma(n+2\frac{m_0\omega}{\hbar}a^2+1)}}. \tag{86}$$

Our main objective was to demonstrate an exact solution of the SE associated with a Hamiltonian using the BenDaniel-Duke kinetic energy operator and a HOP in an external electric field with effective PDM m(x) of type (28). By obtaining analytical wavefunctions and energy spectrum expressions (83) and normalized wavefunctions (84), we were able to do this.

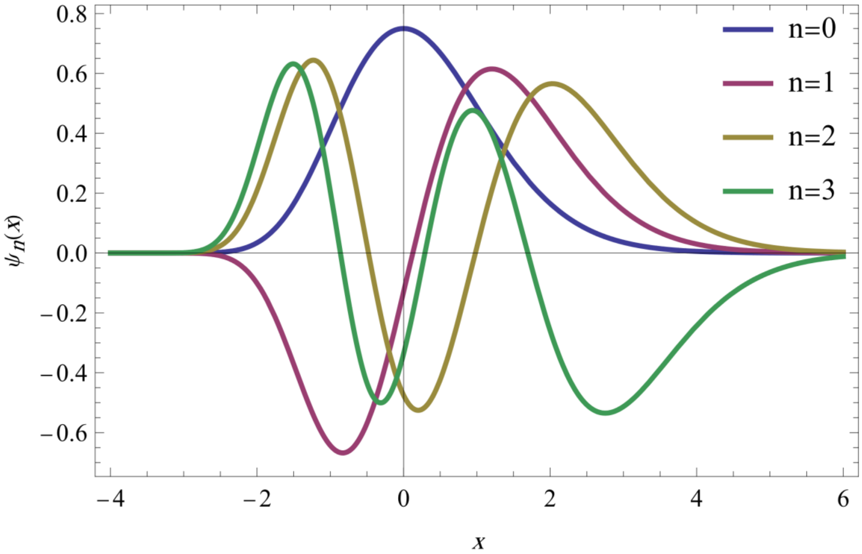

Figure 1. Plot of the Wave functions $\psi_n^{q\varepsilon}(x)$ for $\varepsilon = \frac{1}{2}$, the confinement parameter $a = 4$ ($m_0 = \omega = \hbar = 1$) and $\alpha = 32$

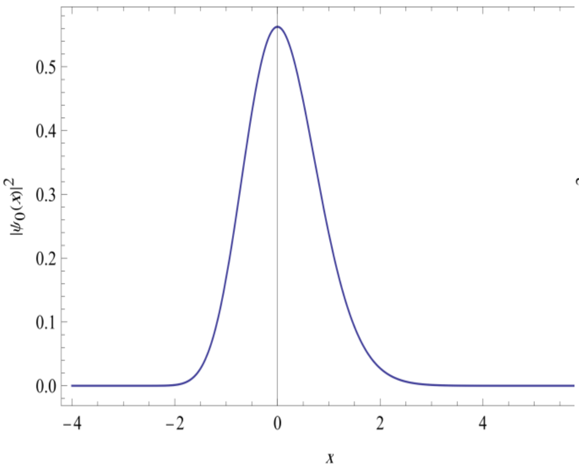

Figure 2.a. $|\psi_0(x)|^2$

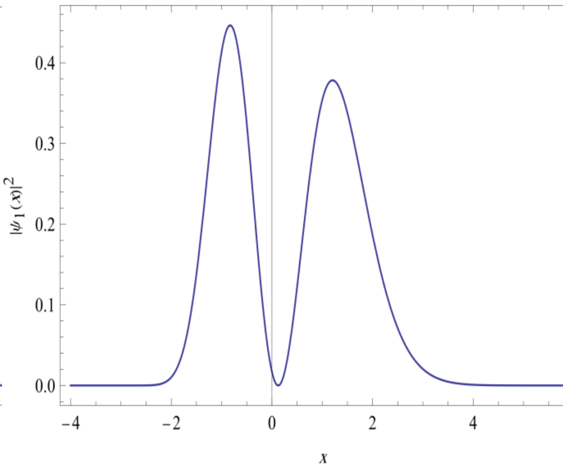

Figure 2.b. $|\psi_1(x)|^2$

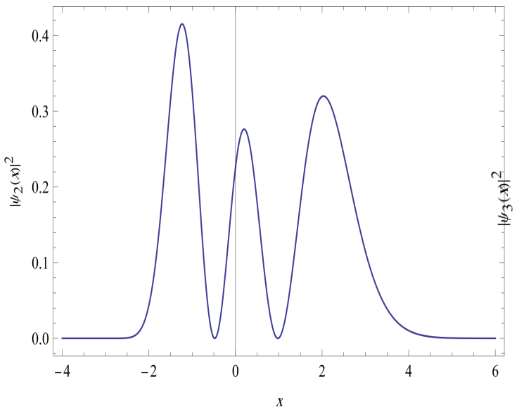

Figure 2.c. $|\psi_2(x)|^2$

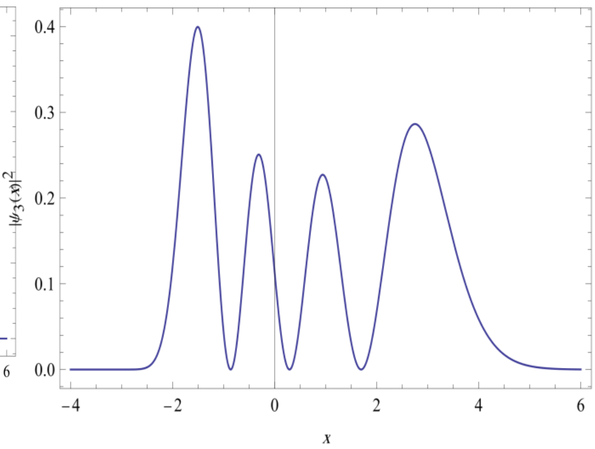

Figure 2.d. $|\psi_3(x)|^2$

Figure 2. Plotting of the probability densities of the wave functions $|\psi_n(x)|^2$ for the value of $\varepsilon = \frac{1}{2}$, the confinement parameter $a = 4$ ($m_0 = \omega = \hbar = 1$) and $\alpha = 32$

Here the wave function was plotted in the plot-range $-4 \leq x \leq 6$ and for the quantum number $0 \leq n \leq 3$, According to the figures, we see that when n rises, new nodes arise in the curve.

Figure 2 shows the probability densities $|\psi_n^{q\varepsilon}(x)|^2$ of the ground state and three excited states at a confinement parameter of $a = 4$.

Consequently, as a quantum number increases, the amplitude decreases and the location moves toward the right side. Also, the numbers of peaks grow as the quantum number (n) increases, as can be seen from our observations.

5. Conclusion

The PDM SE for the most well-known HOP in an external electric field, which has numerous interesting applications such as semiconductors, semiconductor heterostructures, quantum dots and other quantum mechanical systems has been demonstrated using the Nikiforov-Uvarov NU method that proved its powerful efficiency. We investigate a particular expression for the Eigen functions and the energy spectrum. The wave functions of the constructed model are expressed in terms of generalized Laguerre polynomials. Also, it is demonstrated that the mass distribution affects the energy levels of the PDM Schrödinger equation. NU method can be used to find eigenvalues and Eigen functions of Schrödinger type equations. As a consequence, the results have been acquired here, allowing for additional comparisons between the models. The wave functions for the ground state and the other three exited states are displayed explicitly in figures so that one can investigate the bound state.

It shows that as the quantum number increases, new nodes arise; the numbers of peaks increase; the amplitude decreases and the location moves to the right side. The results provide a very well-expected sense.

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